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Existence of Extremal Solutions of Boundary Hemivariational Inequalities

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INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^{0,1}$-boundary $\partial\Omega$, and $\Gamma \subset \partial\Omega$ a relatively open C^1-portion of $\partial\Omega$ having positive surface measure. In this paper we consider the following mixed boundary value problem (BVP for short),

$$Au + Fu = 0 \qquad \text{in} \quad \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on} \quad \partial\Omega \setminus \Gamma, \qquad -\frac{\partial u}{\partial \nu} \in \partial_c j(u) \qquad \text{on} \quad \Gamma, \tag{1.2}$$

where A is a quasilinear elliptic operator in the form

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)),$$

F is the Nemytskij operator of the lower order terms generated by a function $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and defined by

$$Fu(x) = f(x, u(x), \nabla u(x)),$$

and $\partial/\partial\nu$ denotes the outer conormal derivative on Γ related with A. The multivalued boundary condition on Γ of (1.2) is described by Clarke's generalized gradient $\partial_c j\colon \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\varnothing\}$ of a locally Lipschitz function $j\colon \mathbb{R} \to \mathbb{R}$, cf. [4]. Let $\mathscr{V} := W^{1,p}(\Omega)$ be the usual Sobolev space, and let $\mathscr{V}_0 \subset \mathscr{V}$ be the subspace of $\mathscr{V}$ defined by

$$\mathscr{V}_0 = \{u \in \mathscr{V} \mid \gamma u = 0 \text{ on } \partial\Omega \setminus \Gamma\},$$

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where $\gamma\colon \mathcal{V} \to L^p(\partial\Omega)$ is the trace operator which is linear and compact; cf. [10]. (In what follows we keep the notation γ also for its restriction to Γ.)

Under conditions of Leray–Lions type specified later the weak formulation of the BVP (1.1), (1.2) leads to the following hemivariational inequality; cf., e.g., [13, 16].

Find $u \in \mathcal{V}_0$ such that

$$\langle Au + Fu, v - u \rangle + \int_\Gamma j^o(\gamma u; \gamma v - \gamma u)\, d\Gamma \geqslant 0, \qquad \text{for all} \quad v \in \mathcal{V}_0, \tag{1.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{V}_0$ and its dual $\mathcal{V}_0^*$ (respectively $\mathcal{V}$ and $\mathcal{V}^*$), and $j^o(r; s)$ is the generalized directional derivative in the sense of Clarke of the function j at r in the direction s; cf. [4]. If j is, in addition, *regular* in the sense of Clarke to be defined later and satisfies the growth condition

$$\eta \in \partial_c j(s)\colon |\eta| \leqslant c\,(1 + |s|^{p-1}) \qquad \text{for all} \quad s \in \mathbb{R}, \tag{1.4}$$

then the functional $J\colon L^p(\Gamma) \to \mathbb{R}$ defined by

$$J(z) = \int_\Gamma j(z(x))\, d\Gamma, \qquad z \in L^p(\Gamma), \tag{1.5}$$

is locally Lipschitz on $L^p(\Gamma)$ and regular as well, cf. [4]. Moreover, if either j or $-j$ is regular and satisfies (1.4) then the hemivariational inequality (1.3) is equivalent with the following one.

Find $u \in \mathcal{V}_0$ such that

$$\langle Au + Fu, v - u \rangle + J^o(\gamma u; \gamma v - \gamma u) \geqslant 0, \qquad \text{for all} \quad v \in \mathcal{V}_0, \tag{1.6}$$

where $J^o(z; w)$ denotes the generalized directional derivative of J at z in the direction w.

The field of hemivariational inequalities initiated with the pioneering work of Panagiotopoulos (cf., e.g., [14–16]) has attracted increasing attention over the last years mainly due to its many applications in mechanics and engineering. This new type of variational inequalities arise, e.g., in mechanical problems governed by nonconvex, possibly nonsmooth energy functionals (so-called superpotentials), which appear if nonmonotone, multivalued constitutive laws are taken into account. Nonmonotone multivalued boundary conditions of the form (1.2) expressed through the generalized gradient of nonconvex superpotentials stand for certain contact and friction problems (cf., e.g., [13, Sects. 1.4, 3.5, 4.6]) such as adhesive contact at a boundary or an interface, sawtooth contact and friction laws, problems describing deponding and delamination effects and boundary semipermeability problems; cf., e.g., [5, Sects. 3.4, 3.5; 16, Sects. 2.4, 3.5]. Existence and enclosure results for some classes of hemivariational inequalities which are different from that considered here have been obtained recently in [1, 2].

The main goal of this paper is to prove not only existence and enclosure results but the existence of *extremal* solutions for boundary hemivariational inequalities (1.6) or equivalently of BVP (1.1), (1.2) within a sector of an ordered pair of appropriately defined upper and lower solutions when the nonconvex superpotential J is given in the form of a d.c.-functional, which means that J has the representation

$$J(z) = J_1(z) - J_2(z), \tag{1.7}$$

where $J_k\colon L^p(\Gamma) \to \mathbb{R}$, $k = 1, 2$, are convex and continuous. To this end we provide first an equivalent definition for the solution of (1.6) in terms of the subdifferentials of the convex functionals J_k, and introduce a suitable notion of what we call upper and lower solutions which is the basis of our investigations. The nonmonotone and multivalued character of the boundary condition, and a nonlinear elliptic differential operator $A + F$ of (1.1) which is, in general, neither monotone nor coercive are some of the main difficulties in the study of the BVP (1.1), (1.2). This is because there are no comparison results available, and known existence results can be applied only after certain modifications. It should be noted that the results obtained in this paper can be extended to a fully quasilinear operator of the form

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x))$$

which, however, has been omitted in order to emphasize the main idea and to keep technicalities limited.

The plan of the paper is as follows. In Section 2 we introduce the class of d.c.-functionals we are dealing with, provide an equivalent definition of a solution of the hemivariational inequality (1.6) (resp. BVP (1.1), (1.2)), and formulate our main result. In Section 3 the crucial auxiliary lemma is proved which will be used to prove the main result whose proof is given in Section 4. As an application of the theory developed in this paper we treat in Section 5 a boundary semipermeability problem. In this particular application we are even able to prove the existence of global extremal solution in a constructive way without any additional assumptions on the existence of upper and lower solutions.

2. NOTATIONS, HYPOTHESES, AND THE MAIN RESULT

Let us recall first some basic facts from nonsmooth analysis; cf. [4]. To this end let X be real Banach space, X^* its dual space, and let $\Phi: X \to \mathbb{R}$ be a locally Lipschitz functional. The *generalized directional derivative* of Φ at $u \in X$ in the direction $v \in X$, denoted $\Phi^o(u; v)$, is defined as

$$\Phi^o(u; v) = \limsup_{y \to u,\, t \downarrow 0} \frac{\Phi(y + tv) - \Phi(y)}{t},$$

where t is a positive real. It is known that the function $v \to \Phi^o(u; v)$ is finite, convex, positively homogeneous, and subadditive on X, and satisfies $|\Phi^o(u; v)| \leqslant c(\mathscr{U})\, |v|$, where the positive constant $c(\mathscr{U})$ depends only on a neighborhood $\mathscr{U}$ of u. By means of the generalized directional derivative Clarke's *generalized gradient* of Φ at $u \in X$, denoted $\partial_c \Phi(u)$, is defined as the subset of X^* given by

$$\partial_c \Phi(u) = \{\zeta \in X^* \mid \Phi^o(u; v) \geqslant \langle \zeta, v \rangle, \text{ for all } v \in X\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. Since $v \to \Phi^o(u; v)$ is convex on X and satisfies $\Phi^o(u; 0) = 0$, the generalized gradient $\partial_c \Phi(u)$ is nothing but the subdifferential of the functional $v \to \Phi^o(u; v)$ at $v = 0$. The generalized gradient possesses the following properties:

(i) If $\Phi_k: X \to \mathbb{R}$, $k = 1, 2$, are locally Lipschitz then their sum is locally Lipschitz and satisfies for all $u \in X$

$$\partial_c(\Phi_1 + \Phi_2)(u) \subset \partial_c \Phi_1(u) + \partial_c \Phi_2(u); \tag{2.1}$$

(ii) $\partial_c(t\Phi)(u) = t\, \partial_c \Phi(u)$ for any scalar $t \in \mathbb{R}$; (2.2)

(iii) if $\Phi: X \to \mathbb{R}$ is convex then

$$\partial_c \Phi(u) = \partial \Phi(u), \tag{2.3}$$

where ∂ denotes the usual subdifferential of convex functionals.

The *one-sided directional derivative* of Φ at u in the direction v is given by

$$\Phi'(u; v) = \lim_{t \downarrow 0} \frac{\Phi(u + tv) - \Phi(u)}{t}.$$

For instance convex functionals are one-sided directional differentiable. The locally Lipschitz functional Φ is called *regular* at $u \in X$ if for all $v \in X$ the one-sided directional derivative $\Phi'(u; v)$ exists and satisfies

$$\Phi^o(u; v) = \Phi'(u; v) \qquad \text{for all} \quad v \in X.$$

Finally, the functional Φ is *strictly differentiable* at $u \in X$ if there is an element $\zeta \in X^*$ such that for each $v \in X$ one has

$$\lim_{u' \to u,\, t \downarrow 0} \frac{\Phi(u' + tv) - \Phi(u')}{t} = \langle \zeta, v \rangle.$$

We set $D_s\Phi(u) = \zeta$. The following result is due to [4, Proposition 2.2.4].

LEMMA 2.1. *If Φ is strictly differentiable at u, then Φ is Lipschitz near u and $\partial_c\Phi(u) = \{D_s\Phi(u)\}$. Conversely, if Φ is Lipschitz near u and $\partial_c\Phi(u)$ reduces to a singleton $\{\zeta\}$, then Φ is strictly differentiable at u and $D_s\Phi(u) = \zeta$.*

By (2.1) and (2.2) we get for any scalars $t_k \in \mathbb{R}$, $k = 1, 2$,

$$\partial_c(t_1\Phi_1 + t_2\Phi_2)(u) \subset t_1\, \partial_c\Phi_1(u) + t_2\, \partial_c\Phi_2(u), \tag{2.4}$$

and equality holds if one of the Φ_k is strictly differentiable at u; cf. [4, Chap. 2.3, Corollary 2]. Concerning the regularity of locally Lipschitz functionals the following results hold:

(iv) If Φ is strictly differentiable at u then Φ is regular at u.

(v) If Φ: $X \to \mathbb{R}$ is convex then it is regular.

(vi) A finite linear combination (with nonnegative scalars) of functionals regular at u is also regular at u.

Returning to our original BVP (1.1), (1.2), we shall assume the following hypotheses on the function j of the boundary condition and the function f generating the Nemytskij operator F:

(H1) The function j: $\mathbb{R} \to \mathbb{R}$ is locally Lipschitz, satisfies the growth condition (1.4), and is of d.c.-type, i.e., there are convex functions j_k: $\mathbb{R} \to \mathbb{R}$, $k = 1, 2$, such that

$$j(s) = j_1(s) - j_2(s) \qquad \text{for all} \quad s \in \mathbb{R}.$$

(H2) If ∂j_k: $\mathbb{R} \to 2^{\mathbb{R}} \backslash \emptyset$ are the usual subdifferentials of the convex functions j_k then one of the subdifferentials $\partial j_k(s)$ is assumed to be a singleton for each $s \in \mathbb{R}$, and satisfies the growth condition (1.4).

(H3) The function $f: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a *Carathéodory* function, i.e., $f(x, s, \xi)$ is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in (s, ξ) for almost every (a.e.) $x \in \Omega$. There exists a constant $c_0 \geqslant 0$ and a function $k_0 \in L^q(\Omega)$ with $1/p + 1/q = 1$ such that

$$|f(x, s, \xi)| \leqslant k_0(x) + c_0 \left(|s|^{p-1} + |\xi|^{p-1}\right), \tag{2.5}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

Further we impose the following standard conditions of Leray-Lions type on the coefficient functions $a_i: \Omega \times \mathbb{R}^N \to \mathbb{R}$, $i = 1, ..., N$.

(A1) Each $a_i(x, \xi)$ is a Carathéodory function, i.e., measurable in $x \in \Omega$ for all $\xi \in \mathbb{R}^N$ and continuous in $\xi \in \mathbb{R}^N$ for a.e. $x \in \Omega$. There exists a constant $c_1 > 0$ and a function $k_1 \in L^q(\Omega)$ such that

$$|a_i(x, \xi)| \leqslant k_1(x) + c_1 |\xi|^{p-1},$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$.

(A2) $\sum_{i=1}^N (a_i(x, \xi) - a_i(x, \xi'))(\xi_i - \xi_i') > 0$ for a.e. $x \in \Omega$ and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(A3) There exists a positive constant μ and a function $k_2 \in L^1(\Omega)$ such that

$$\sum_{i=1}^N a_i(x, \xi)\, \xi_i \geqslant \mu\, |\xi|^p + k_2(x), \qquad \text{for a.e.} \quad x \in \Omega \qquad \text{and for all} \quad \xi \in \mathbb{R}^N.$$

We introduce the natural partial ordering in $L^p(\Omega)$, that is $u \leqslant w$ if and only if $w - u$ belongs to the set $L^p_+(\Omega)$ of all nonnegative elements of $L^p(\Omega)$, which induces also a partial ordering in the Sobolev space $\mathscr{V}$. If $u, w \in \mathscr{V}$ and $u \leqslant w$ then

$$[u, w] = \{v \in \mathscr{V} \mid u \leqslant v \leqslant w\}$$

denotes the order interval formed by u and w.

Let a denote the semilinear form associated with the differential operator A by

$$\langle Au, \varphi \rangle := a(u, \varphi) = \int_\Omega \sum_{i=1}^N a_i(x, \nabla u) \frac{\partial \varphi}{\partial x_i}\, dx.$$

Then by hypotheses (A1) and (A2) the operator $A: \mathscr{V} \to \mathscr{V}^*$ (resp. $A: \mathscr{V}_0 \to \mathscr{V}_0^*$) is continuous and bounded, and by hypothesis (H3) the Nemytskij operator $F: \mathscr{V} \to L^q(\Omega) \subset \mathscr{V}^*$ is continuous and bounded as well. Taking the regularity properties (iv), (v), and (vi) and Lemma 2.1 into account from hypotheses (H1), (H2) we conclude that either j or $-j$ must be regular in the sense of Clarke defined above. Furthermore, the generalized subgradient of j is given by

$$\partial_c j(s) = \partial j_1(s) - \partial j_2(s) \qquad \text{for all} \quad s \in \mathbb{R}. \tag{2.6}$$

This is because convexity of $j_k\colon \mathbb{R} \to \mathbb{R}$ implies that j_k, $k=1, 2$, are locally Lipschitzian, and thus from Lemma 2.1 and (H2) it follows that for each $s \in \mathbb{R}$ one of the functions j_k must be strictly differentiable in s. Hence from generalized subdifferential calculus given above we get

$$\partial_c j(s) = \partial_c j_1(s) - \partial_c j_2(s)$$

which yields in view of the convexity of the j_k the relation (2.6). Moreover, by (H1) and (H2) it follows that the functional $J\colon L^p(\Gamma) \to \mathbb{R}$ defined by (1.5) is locally Lipschitz, and since either j or $-j$ is regular, we get in both cases

$$\int_\Gamma j^o(\gamma u; \gamma v - \gamma u)\, d\Gamma = J^o(\gamma u; \gamma v - \gamma u),$$

which shows that (1.3) and (1.6) are equivalent. Hence in what follows we may consider either (1.3) or (1.6) as the corresponding equivalent weak formulation of our original BVP (1.1), (1.2).

Next we provide an equivalent characterization of problem (1.6) which will be used in our investigations. To this end let us define the multivalued mapping $\partial_c^\gamma J\colon \mathscr{V}_0 \to 2^{\mathscr{V}_0^*}$ by the formula

$$\partial_c^\gamma J(u) := \{\zeta \in \mathscr{V}_0^* \mid J^o(\gamma u; \gamma v) \geqslant \langle \zeta, v\rangle \text{ for all } v \in \mathscr{V}_0\}. \tag{2.7}$$

Then problem (1.6) can equivalently be written as the following inclusion in $\mathscr{V}_0^*$:

$$-Au - Fu \in \partial_c^\gamma J(u). \tag{2.8}$$

The regularity of either j or $-j$ implies the regularity of the integral functional J or $-J$, respectively (cf. [4]), which in either case yields the equality, cf. [13],

$$J^o(\gamma u; \gamma v) = (J \circ \gamma)^o\,(u; v)$$

and thus the generalized gradient $\partial_c(J \circ \gamma)$ of the composition $J \circ \gamma$ satisfies

$$\partial_c(J \circ \gamma)(u) = \partial_c^\gamma J(u) \qquad \text{for all} \quad u \in \mathscr{V}_0. \tag{2.9}$$

Since $J\colon L^p(\Gamma)\to\mathbb{R}$ is locally Lipschitz and $\gamma\colon \mathscr{V}_0\to L^p(\Gamma)$ is linear and compact we may apply the chain rule (cf. [4, Theorem 2.3.10]) to calculate the generalized gradient of the composition which yields in view of the regularity of either J or $-J$ the result

$$\partial_c(J\circ\gamma)(u)=\gamma^*\circ\partial_c J(\gamma u)\qquad\text{for all}\quad u\in\mathscr{V}_0,\tag{2.10}$$

where $\gamma^*\colon L^q(\Gamma)\to\mathscr{V}_0^*$ denotes the adjoint operator of γ given by

$$\gamma^* w(v)=\int_\Gamma w(x)\,\gamma v(x)\,d\Gamma,\qquad v\in\mathscr{V}_0,\tag{2.11}$$

for $w\in L^q(\Gamma)$. Further, Hypotheses (H1) and (H2) allow us to apply [4, Theorem 2.7.5] which yields due to the regularity of either j or $-j$ for the generalized gradient of the integral functional $J\colon L^p(\Gamma)\to\mathbb{R}$

$$\partial_c J(z)=\int_\Gamma \partial_c j(z(x))\,d\Gamma\qquad\text{for all}\quad z\in L^p(\Gamma),\tag{2.12}$$

which means that $w\in\partial_c J(z)$ if and only if $w\in L^p(\Gamma)^*=L^q(\Gamma)$ and $w(x)\in\partial_c j(z(x))$ a.e. in Γ. In view of (2.10), (2.11), and (2.12) we have the following characterization of $\partial_c(J\circ\gamma)(u)$ for each $u\in\mathscr{V}_0$: $\zeta\in\partial_c(J\circ\gamma)(u)$ if and only if there is a $w\in L^q(\Gamma)$ such that $w(x)\in\partial_c j(\gamma u(x))$ for a.e. $x\in\Gamma$ and

$$\langle\zeta, v\rangle=\int_\Gamma w(x)\,\gamma v(x)\,d\Gamma\qquad\text{for all}\quad v\in\mathscr{V}_0.\tag{2.13}$$

Let $J_k\colon L^p(\Gamma)\to\mathbb{R}$ be the integral functionals associated with the convex functions $j_k\colon\mathbb{R}\to\mathbb{R}$. Then by Hypotheses (H1) and (H2) these functionals are convex and satisfy

$$J(z)=J_1(z)-J_2(z)\qquad\text{for all}\quad z\in L^p(\Gamma),$$

and, moreover, the equality (2.6) holds accordingly, i.e.,

$$\partial_c J(z)=\partial J_1(z)-\partial J_2(z)\qquad\text{for all}\quad z\in L^p(\Gamma),\tag{2.14}$$

where ∂J_k, $k=1,2$, denote the usual subdifferentials of convex functionals. That is, $w\in\partial_c J(z)$ if and only if there are $w_k\in L^q(\Gamma)$ such that $w_k\in\partial J_k(z)$, $k=1,2$, $w=w_1-w_2$ and $w_k(x)\in\partial j_k(z(x))$ a.e. in Γ, where in view of hypothesis (H2) the w_k are uniquely defined. Since the hemivariational inequality (1.6) is equivalent with the inclusion (2.8) which by (2.9) is the same as

$$-Au-Fu\in\partial_c(J\circ\gamma)(u),\qquad u\in\mathscr{V}_0,\tag{2.15}$$

we obtain by means of (2.10)–(2.14) the following equivalent definition of a solution of (1.6).

DEFINITION 2.1. A function $u \in \mathscr{V}_0$ is a *solution* of the hemivariational inequality (1.6) if there are $w_1, w_2 \in L^q(\Gamma)$ such that

(i) $w_k(x) \in \partial j_k(\gamma u(x))$, $k=1, 2$, for a.e. $x \in \Gamma$, and
(ii) $\langle Au + Fu, v \rangle + \int_\Gamma w_1\, \gamma v\, d\Gamma = \int_\Gamma w_2\, \gamma v\, d\Gamma$ for all $v \in \mathscr{V}_0$.

It is well known that the subdifferentials $\partial j_k \colon \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ can be characterized by nondecreasing functions $h_k \colon \mathbb{R} \to \mathbb{R}$ in the following way:

$$\partial j_k(s) = [h_k(s-), h_k(s+)],$$

where the $h_k(s\pm)$ denote the one-sided limits given by $h_k(s\pm) := \lim_{\varepsilon \downarrow 0} h_k(s \pm \varepsilon)$, $k=1, 2$. Denote by $\bar{h}_k, \underline{h}_k \colon \mathbb{R} \to \mathbb{R}$ the one-sided limits, i.e., $\bar{h}_k(s) := h_k(s+)$, $\underline{h}_k(s) := h_k(s-)$ and let $\bar{H}_k$ and $\underline{H}_k$ be the corresponding Nemytskij operators related with $\bar{h}_k$ and $\underline{h}_k$, $k=1, 2$, respectively. Now we introduce the notion of upper and lower solution for the hemivariational inequality (1.6).

DEFINITION 2.2. A function $\bar{u} \in \mathscr{V}$ is called an *upper solution* of the hemivariational inequality (1.6) if there is a $\bar{w}_1 \in L^q(\Gamma)$ such that

(i) $\bar{u} \geqslant 0$ on $\partial\Omega \setminus \Gamma$,
(ii) $\bar{w}_1(x) \in \partial j_1(\gamma \bar{u}(x))$, for a.e. $x \in \Gamma$, and
(iii) $\langle A\bar{u} + F\bar{u}, v \rangle + \int_\Gamma \bar{w}_1\, \gamma v\, d\Gamma \geqslant \int_\Gamma \bar{H}_2(\gamma \bar{u})\, \gamma v\, d\Gamma$ for all $v \in \mathscr{V}_0 \cap L^p_+(\Omega)$.

DEFINITION 2.3. A function $\underline{u} \in \mathscr{V}$ is called a *lower solution* of the hemivariational inequality (1.6) if there is a $\underline{w}_1 \in L^q(\Gamma)$ such that

(i) $\underline{u} \leqslant 0$ on $\partial\Omega \setminus \Gamma$,
(ii) $\underline{w}_1(x) \in \partial j_1(\gamma \underline{u}(x))$, for a.e. $x \in \Gamma$, and
(iii) $\langle A\underline{u} + F\underline{u}, v \rangle + \int_\Gamma \underline{w}_1\, \gamma v\, d\Gamma \leqslant \int_\Gamma \underline{H}_2(\gamma \underline{u})\, \gamma v\, d\Gamma$ for all $v \in \mathscr{V}_0 \cap L^p_+(\Omega)$.

Finally, we define the notion of extremal solutions with respect to an order interval.

DEFINITION 2.4. A solution u^* is called the *greatest solution* within the order interval $[\underline{u}, \bar{u}]$ if for any solution $u \in [\underline{u}, \bar{u}]$ we have $u \leqslant u^*$. Similarly, u_* is the *least solution* in $[\underline{u}, \bar{u}]$ if for any solution $u \in [\underline{u}, \bar{u}]$ it holds $u_* \leqslant u$. The least and greatest solutions are called the *extremal* ones within the sector $[\underline{u}, \bar{u}]$.

The main result of this paper asserts the existence of extremal solutions of the hemivariational inequality (1.6) (resp. BVP (1.1), (1.2)) within an order interval $[\underline{u}, \bar{u}]$ of upper and lower solutions and is given in the following theorem.

THEOREM 2.1. *Let the Hypotheses* (A1)–(A3) *and* (H1)–(H3) *be satisfied and assume the existence of upper and lower solutions $\bar{u}$ and $\underline{u}$, respectively, of the hemivariational inequality* (1.6) *satisfying $\underline{u} \leqslant \bar{u}$. Then there exist extremal solutions within the order interval $[\underline{u}, \bar{u}]$.*

In the proof of Theorem 2.1 we focus on the existence of the greatest solution only, since the existence of the least solution can be shown in a similar way.

3. PRELIMINARIES

Throughout this section we shall assume that all the hypotheses of Theorem 2.1 are satisfied and that $\bar{u}$ and $\underline{u}$ are upper and lower solutions satisfying $\underline{u} \leqslant \bar{u}$.

We consider first the following variational inequality.

Find $u \in \mathcal{V}_0$ such that there is a $w_1 \in L^q(\Gamma)$ satisfying $w_1(x) \in \partial j_1(\gamma u(x))$ a.e. on Γ and

$$\langle Au + Fu, \varphi \rangle + \int_\Gamma w_1 \, \gamma\varphi \, d\Gamma = \int_\Gamma \bar{H}_2(\gamma \bar{u}) \, \gamma\varphi \, d\Gamma, \tag{3.1}$$

for all $\varphi \in \mathcal{V}_0$, where $\bar{H}_2(\gamma \bar{u}) \in L^q(\Gamma)$. Define a functional h by

$$\langle h, \varphi \rangle = \int_\Gamma \bar{H}_2(\gamma \bar{u}) \, \gamma\varphi \, d\Gamma \qquad \text{for all} \quad \varphi \in \mathcal{V}_0.$$

Then $h \in \mathcal{V}_0^*$ and (3.1) is equivalent with

$$u \in \mathcal{V}_0: \quad Au + Fu + \partial J_1(\gamma u) \ni h. \tag{3.2}$$

We are going to prove the following lemma.

LEMMA 3.1. *The variational inequality* (3.1) (*resp.* (3.2)) *possesses extremal solutions within the order interval formed by the given upper and lower solutions $\bar{u}$ and $\underline{u}$ of the original hemivariational inequality, respectively.*

Proof. The proof will be carried out in several steps and is designed to show the existence of the greatest solution u^* only, since the existence of the least solution u_* can be proved in a similar way.

(a) Existence of solutions in $[\underline{u}, \bar{u}]$. Hypotheses (A1)–(A3) and (H3) imply that the operator $A+F\colon \mathscr{V}_0 \to \mathscr{V}_0^*$ is of Leray-Lions type, i.e., it is continuous, bounded and pseudomonotone, cf., e.g., [17, Chap. II, Theorem 6.1]. The trace operator $\gamma\colon \mathscr{V}_0 \to L^p(\Gamma)$ is linear and compact, and the integral functional $J_1\colon L^p(\Gamma) \to \mathbb{R}$ generated by the convex function $j_1\colon \mathbb{R} \to \mathbb{R}$ is convex and continuous due to (H2). Thus the composition $J_1 \circ \gamma\colon \mathscr{V}_0 \to \mathbb{R}$ is convex and continuous on the whole space. By applying a known existence result for variational inequalities involving pseudomonotone operators such as, e.g., [18, Theorem 54.A] problem (3.1) (resp. (3.2)) admits a solution provided the operator $A+F$ is, in addition, coercive. Unfortunately this last condition may fail. To overcome this difficulty we consider instead of (3.1) the following associated truncated problem which takes into account that we are only interested in solutions within an order interval:

$$u \in \mathscr{V}_0 : Au + F \circ Tu + \lambda Bu + \partial J_1(\gamma u) \ni h, \tag{3.3}$$

where $\lambda > 0$ is some constant to be specified later, and B is the Nemytskij operator generated by the cut-off function $b\colon \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$b(x,s) = \begin{cases} (s - \bar{u}(x))^{p-1} & \text{if } \ s > \bar{u}(x), \\ 0 & \text{if } \ \underline{u}(x) \leqslant s \leqslant \bar{u}(x), \\ -(\underline{u}(x) - s)^{p-1} & \text{if } \ s < \underline{u}(x), \end{cases}$$

and T is the truncation operator defined by

$$Tu(x) = \begin{cases} \bar{u}(x) & \text{if } \ u(x) > \bar{u}(x), \\ u(x) & \text{if } \ \underline{u}(x) \leqslant u(x) \leqslant \bar{u}(x), \\ \underline{u}(x) & \text{if } \ u(x) < \underline{u}(x). \end{cases}$$

It is well known that the operator $T\colon \mathscr{V} \to \mathscr{V}$ is bounded and continuous (cf. [6]) which implies by (H3) that the composed operator $F \circ T$: $\mathscr{V} \to L^q(\Omega)$ is bounded and continuous as well. The function b is a Carathéodory function satisfying a growth condition of the form

$$|b(x,s)| \leqslant k_3(x) + c_2\, |s|^{p-1} \tag{3.4}$$

for some positive constant c_2 and some function $k_3 \in L^q(\Omega)$, as well as

$$\int_\Omega b(x, u(x))\, u(x)\, dx \geqslant c_3\, \|u\|^p_{L^p(\Omega)} - c_4 \tag{3.5}$$

for some positive constants c_3, c_4. By (3.4) it follows that the Nemytskij operator B associated with the function b is bounded and continuous from $L^p(\Omega)$ into $L^q(\Omega) \subset \mathscr{V}_0^*$, and thus by the compact embedding $\mathscr{V}_0 \subset L^p(\Omega)$ the operator $B: \mathscr{V}_0 \to \mathscr{V}_0^*$ is compact. Hence it follows that the operator $A + F \circ T + \lambda B: \mathscr{V}_0 \to \mathscr{V}_0^*$ is bounded, continuous and pseudomonotone. To verify the coercivity of $A + F \circ T + \lambda B$ we have to show that

$$\frac{\langle (A + F \circ T + \lambda B)\, u, u \rangle}{\|u\|_{\mathscr{V}_0}} \to \infty \qquad \text{as} \quad \|u\|_{\mathscr{V}_0} \to \infty. \tag{3.6}$$

Hypothesis (A3) implies

$$\langle Au, u \rangle \geqslant \mu \, \|\nabla u\|^p_{L^p(\Omega)} - c, \qquad \text{for some} \quad c > 0, \tag{3.7}$$

and by means of (H3) and applying Young's inequality we obtain for any $\varepsilon > 0$ an estimate of the form

$$\begin{aligned} |\langle F \circ Tu, u \rangle| &= \left| \int_\Omega f(\cdot, Tu, \nabla Tu)\, u \, dx \right| \\ &\leqslant \varepsilon \, \|\nabla u\|^p_{L^p(\Omega)} + c(\varepsilon) \, \|u\|^p_{L^p(\Omega)} + c \, \|u\|_{L^p(\Omega)}, \end{aligned} \tag{3.8}$$

where $c(\varepsilon)$ is some constant depending only on ε. Choosing $\varepsilon < \mu$ the coercivity of $A + F \circ T + \lambda B$ follows from (3.5), (3.7), and (3.8) by taking λ sufficiently large, which finally ensures the existence of solutions of the variational inequality (3.3).

Next we are going to show that any solution of (3.3) belongs to the interval $[\underline{u}, \bar{u}]$. To this end let u be any solution of (3.3) which means

$$u \in \mathscr{V}_0: \langle Au + F \circ Tu + \lambda Bu, \varphi \rangle + \int_\Gamma w_1 \, \gamma u \, d\Gamma = \langle h, \varphi \rangle, \qquad \varphi \in \mathscr{V}_0, \tag{3.9}$$

where $w_1 \in L^q(\Gamma)$ satisfies $w_1(x) \in \partial j_1(\gamma u(x))$ for a.e. $x \in \Gamma$. According to Definition 2.2 the upper solution satisfies $\bar{u} \geqslant 0$ on $\partial\Omega \setminus \Gamma$, and

$$\langle A\bar{u} + F\bar{u}, v \rangle + \int_\Gamma \bar{w}_1 \, \gamma v \, d\Gamma \geqslant \int_\Gamma \bar{H}_2(\gamma \bar{u}) \, \gamma v \, d\Gamma = \langle h, v \rangle \tag{3.10}$$

for all $v \in \mathscr{V}_0 \cap L^p_+(\Omega)$, where $\bar{w}_1(x) \in \partial j_1(\gamma \bar{u}(x))$, for a.e. $x \in \Gamma$. Subtracting (3.10) from (3.9) we obtain for all $\varphi \in \mathscr{V}_0 \cap L^p_+(\Omega)$ the inequality

$$\langle Au - A\bar{u} + F \circ Tu - F\bar{u} + \lambda Bu, \varphi \rangle + \int_\Gamma (w_1 - \bar{w}_1) \, \gamma\varphi \, d\Gamma \leqslant 0. \tag{3.11}$$

In particular an admissible test function in (3.11) is $\varphi = (u - \bar{u})^+$ where $w^+ := \max(w, 0)$, cf., e.g., [8]. By (A2) we get

$$\langle Au - A\bar{u}, (u-\bar{u})^+ \rangle = a(u, (u-\bar{u})^+) - a(\bar{u}, (u-\bar{u})^+)$$
$$= \int_{\{u>\bar{u}\}} \sum_{i=1}^{N} (a_i(x, \nabla u) - a_i(x, \nabla \bar{u})) \frac{\partial(u-\bar{u})}{\partial x_i} \, dx \geqslant 0, \tag{3.12}$$

and the property of the truncation operator results in

$$\langle F \circ Tu - F\bar{u}, (u-\bar{u})^+ \rangle = \int_\Omega (f(\cdot, Tu, \nabla Tu) - f(\cdot, \bar{u}, \nabla \bar{u}))(u-\bar{u})^+ \, dx = 0. \tag{3.13}$$

The maximal monotonicity of the subdifferential ∂j_1 implies

$$\int_\Gamma (w_1 - \bar{w}_1)\, \gamma(u-\bar{u})^+ \, d\Gamma \geqslant 0. \tag{3.14}$$

In view of (3.12), (3.13), and (3.14) we obtain from (3.11) the inequality

$$0 \geqslant \lambda \langle Bu, (u-\bar{u})^+ \rangle = \lambda \int_\Omega Bu(u-\bar{u})^+ \, dx$$
$$= \lambda \int_{\{u>\bar{u}\}} (u-\bar{u})^{p-1} (u-\bar{u}) \, dx \geqslant 0,$$

which shows that $((u-\bar{u})^+)^p = 0$ and thus $u \leqslant \bar{u}$ a.e. in Ω. To show that the inequality $\underline{u} \leqslant u$ is valid we recall the conditions satisfied by $\underline{u}$. The lower solution $\underline{u}$ satisfies $\underline{u} \leqslant 0$ on $\partial\Omega \setminus \Gamma$, and

$$\langle A\underline{u} + F\underline{u}, v \rangle + \int_\Gamma \underline{w}_1 \, \gamma v \, d\Gamma \leqslant \int_\Gamma \underline{H}_2(\gamma\underline{u}) \, \gamma v \, d\Gamma \tag{3.15}$$

for all $v \in \mathscr{V}_0 \cap L^p_+(\Omega)$, where $\underline{w}_1(x) \in \partial j_1(\gamma\underline{u}(x))$, for a.e. $x \in \Gamma$. Subtracting (3.9) from (3.15) and taking into account

$$\int_\Gamma \underline{H}_2(\gamma\underline{u}) \, \gamma\varphi \, d\Gamma \leqslant \int_\Gamma \bar{H}_2(\gamma\bar{u}) \, \gamma\varphi \, d\Gamma = \langle h, \varphi \rangle \qquad \text{for all} \quad \varphi \in \mathscr{V}_0 \cap L^p_+(\Omega)$$

we obtain for all $\varphi \in \mathscr{V}_0 \cap L^p_+(\Omega)$

$$\langle A\underline{u} - Au + F\underline{u} - F \circ Tu - \lambda Bu, \varphi \rangle + \int_\Gamma (\underline{w}_1 - w_1) \, \gamma\varphi \, d\Gamma \leqslant 0. \tag{3.16}$$

Taking, in particular, the test function $\varphi = (\underline{u} - u)^+$ in (3.16) and using similar arguments as before we end up with the inequality

$$-\lambda \int_\Omega Bu(\underline{u} - u)^+ \, dx \leqslant 0. \tag{3.17}$$

By definition of the operator B from (3.17) we get

$$0 \leqslant \int_\Omega Bu(\underline{u} - u)^+ \, dx = \int_{\{\underline{u} > u\}} -(\underline{u} - u)^{p-1} (\underline{u} - u) \, dx$$

$$= -\int_\Omega ((\underline{u} - u)^+)^p \, dx \leqslant 0,$$

which yields $(\underline{u} - u)^+ = 0$, and thus $\underline{u} \leqslant u$ a.e. in Ω. This proves that any solution of the auxiliary problem (3.3) is in the interval $[\underline{u}, \bar{u}]$, which implies $Tu = u$ and $Bu = 0$, and hence it follows that any solution of (3.3) must be a solution of (3.1) (resp. (3.2)) within $[\underline{u}, \bar{u}]$ which completes the existence part of Lemma 3.1.

Let $\mathscr{S} \neq \varnothing$ denote the set of all solutions of (3.1) (resp. (3.2)) within $[\underline{u}, \bar{u}]$.

(b) $\mathscr{S}$ is upward directed. Here we are going to show that $\mathscr{S}$ is an *upward directed* set, that is, $\mathscr{S}$ has the property that whenever $u_1, u_2 \in \mathscr{S}$ then there is an element $u_3 \in \mathscr{S}$ such that $u_1 \leqslant u_3$ and $u_2 \leqslant u_3$.

Let $\hat{u} = \max(u_1, u_2)$ and define $\hat{w}$ by

$$\hat{w}(x) = \begin{cases} w_{1,1}(x) & \text{if} \quad x \in \{\gamma u_1 \geqslant \gamma u_2\}, \\ w_{1,2}(x) & \text{if} \quad x \in \{\gamma u_2 > \gamma u_1\}, \end{cases}$$

where $u_k \in \mathscr{S}$, $k = 1, 2$, satisfy

$$\langle Au_k + Fu_k, \varphi \rangle + \int_\Gamma w_{1,k} \gamma\varphi \, d\Gamma = \langle h, \varphi \rangle, \qquad \varphi \in \mathscr{V}_0, \tag{3.18}$$

with $w_{1,k} \in L^q(\Gamma)$ and $w_{1,k}(x) \in \partial j_1(\gamma u_k(x))$ for a.e. $x \in \Gamma$. Obviously $\hat{w} \in L^q(\Gamma)$ and $\hat{w}(x) \in \partial j_1(\gamma \hat{u}(x))$ for a.e. $x \in \Gamma$. Next we shall show that $\hat{u}$ satisfies

$$\langle A\hat{u} + F\hat{u}, \varphi \rangle + \int_\Gamma \hat{w} \, \gamma\varphi \, d\Gamma \leqslant \langle h, \varphi \rangle, \qquad \text{for all} \quad \varphi \in \mathscr{V}_0 \cap L^p_+(\Omega), \tag{3.19}$$

which proves that $\hat{u}$ is a lower solution of (3.1). To this end let $\varrho_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a monotone nondecreasing smooth function for any $\varepsilon > 0$ satisfying

$$\varrho_\varepsilon(s) = 0 \quad \text{for} \quad s \leqslant 0, \qquad \varrho_\varepsilon(s) = 1 \quad \text{for} \quad s \geqslant \varepsilon,$$
$$\text{and} \qquad \varrho_\varepsilon(s) \to \chi_{\{s>0\}} \quad \text{as} \quad \varepsilon \to 0,$$

where $\chi_{\{s>0\}}$ denotes the characteristic function of the positive real line $\mathbb{R}_{>0}$. Let us introduce the following set of smooth functions

$$\mathscr{D}_\Gamma^+ := \{\psi \in C^\infty(\Omega) \cap \mathscr{V} \mid \psi \geqslant 0 \text{ and } \psi = 0 \text{ on } \partial\Omega \setminus \Gamma\},$$

whose closure in $\mathscr{V}$ coincides with $\mathscr{V}_0 \cap L_+^p(\Omega)$. Let $\psi \in \mathscr{D}_\Gamma^+$. Taking as special (nonnegative) test function in (3.18) for $k=1$ the function $\varphi = (1 - \varrho_\varepsilon(u_2 - u_1))\, \psi \in \mathscr{V}_0 \cap L_+^p(\Omega)$ and for $k=2$ the function $\varphi = \varrho_\varepsilon(u_2 - u_1)\, \psi \in \mathscr{V}_0 \cap L_+^p(\Omega)$ and adding the resulting equations we obtain

$$\begin{aligned} &\langle Au_1 + Fu_1, \psi \rangle + \langle Au_2 - Au_1 + Fu_2 - Fu_1, \varrho_\varepsilon(u_2 - u_1)\, \psi \rangle \\ &\quad + \int_\Gamma \{w_{1,1} \gamma\psi + (w_{1,2} - w_{1,1})\, \gamma(\varrho_\varepsilon(u_2 - u_1)\, \psi)\}\, d\Gamma = \langle h, \psi \rangle. \end{aligned} \tag{3.20}$$

The partial derivative of $\varrho_\varepsilon(u_2 - u_1)\, \psi$ yields

$$\frac{\partial}{\partial x_i} (\varrho_\varepsilon(u_2 - u_1)\, \psi) = \psi \varrho'_\varepsilon(u_2 - u_1) \frac{\partial(u_2 - u_1)}{\partial x_i} + \varrho_\varepsilon(u_2 - u_1) \frac{\partial \psi}{\partial x_i},$$

which gives in view of (A2)

$$\begin{aligned} &\langle Au_2 - Au_1, \varrho_\varepsilon(u_2 - u_1)\, \psi \rangle \\ &\quad = a(u_2, \varrho_\varepsilon(u_2 - u_1)\, \psi) - a(u_1, \varrho_\varepsilon(u_2 - u_1)\, \psi) \\ &\quad = \int_\Omega \sum_{i=1}^N \Bigg(a_i(\cdot, \nabla u_2) - a_i(\cdot, \nabla u_1))(\psi \varrho'_\varepsilon(u_2 - u_1) \\ &\qquad \times \frac{\partial(u_2 - u_1)}{\partial x_i} + \varrho_\varepsilon(u_2 - u_1) \frac{\partial \psi}{\partial x_i} \Bigg)\, dx \\ &\quad \geqslant \int_\Omega \sum_{i=1}^N (a_i(\cdot, \nabla u_2) - a_i(\cdot, \nabla u_1))\, \varrho_\varepsilon(u_2 - u_1) \frac{\partial \psi}{\partial x_i}\, dx. \end{aligned} \tag{3.21}$$

Applying Lebesgue's dominated convergence theorem and taking into account that

$$\varrho_\varepsilon(u_2 - u_1) \to \chi_{\{u_2 > u_1\}} \quad \text{as} \quad \varepsilon \to 0,$$

from (3.20) and (3.21) we obtain the inequality

$$\int_\Omega \sum_{i=1}^N \left\{ a_i(\cdot, \nabla u_1) \frac{\partial \psi}{\partial x_i} + (a_i(\cdot, \nabla u_2) - a_i(\cdot, \nabla u_1)) \chi_{\{u_2 > u_1\}} \frac{\partial \psi}{\partial x_i} \right\} dx$$

$$+ \int_\Omega \{Fu_1 \psi + (Fu_2 - Fu_1) \chi_{\{u_2 > u_1\}} \psi\} \, dx$$

$$+ \int_\Gamma \{w_{1,1} \gamma \psi + (w_{1,2} - w_{1,1}) \chi_{\{\gamma u_2 > \gamma u_1\}} \gamma \psi\} \, d\Gamma \leqslant \langle h, \psi \rangle, \tag{3.22}$$

which holds for all $\psi \in \mathscr{D}_\Gamma^+$. By completion (3.22) holds likewise for any $\varphi \in \mathscr{V}_0 \cap L_+^p(\Omega)$ which is nothing else as (3.19), and thus $\hat{u} \in [\underline{u}, \bar{u}]$ is a lower solution of (3.1). By using a cut-off function and a truncation mapping with respect to the interval $[\hat{u}, \bar{u}]$ and applying the result of part (a) there exist solutions of the variational inequality (3.1) within the interval $[\hat{u}, \bar{u}]$ which proves that the solution set $\mathscr{S}$ is upward directed.

(c) $\mathscr{S}$ satisfies Zorn's Lemma. We are going to verify the supposition of Zorn's Lemma; i.e., we have to show that any well-ordered chain $\mathscr{C} \subset \mathscr{S}$ possesses an upper bound in $\mathscr{S}$.

To this end we first show that the solution set $\mathscr{S}$ is bounded in $\mathscr{V}_0$, i.e.,

$$\|u\|_{\mathscr{V}_0} \leqslant c \qquad \text{for all} \quad u \in \mathscr{S}. \tag{3.23}$$

By using the special test function $\varphi = u$ and taking into account the growth conditions for f and ∂j_k as well as the monotonicity of ∂j_k, and applying Young's inequality the left-hand side of (3.1) can be estimated below as

$$\langle Au + Fu, u \rangle + \int_\Gamma w_1 \gamma u \, d\Gamma$$

$$\geqslant \mu \|\nabla u\|_{L^p(\Omega)}^p - \|k_2\|_{L_1(\Omega)} - \|k_0\|_{L^q(\Omega)} \|u\|_{L^p(\Omega)} - c_0 \|u\|_{L^p(\Omega)}^p$$

$$- \delta_1 \|\nabla u\|_{L^p(\Omega)}^p - c(\delta_1) \|u\|_{L^p(\Omega)}^p - c(\delta_2) - \delta_2 \int_\Gamma |\gamma u|^p \, d\Gamma, \tag{3.24}$$

where δ_1, δ_2 may be any positive constants to be chosen appropriately and $c > 0$ is some general constant not depending on u. The right-hand side of (3.1) yields an estimate of the form

$$|\langle h, u \rangle| \leqslant \|h\|_{\mathscr{V}_0^*} \|u\|_{\mathscr{V}_0} \leqslant c(\delta_3) + \delta_3 \|u\|_{\mathscr{V}_0}^p, \tag{3.25}$$

for any $\delta_3 > 0$. The trace operator $\gamma: \mathscr{V} \to L^p(\partial\Omega)$ is linear and continuous (even compact) which yields

$$\int_\Gamma |\gamma u|^p \, d\Gamma \leqslant c \, \|u\|^p_{\mathscr{V}_0}. \tag{3.26}$$

As the solution set $\mathscr{S} \subset [\underline{u}, \bar{u}]$ is uniformly $L^p(\Omega)$-bounded, its boundedness with respect to the norm of $\mathscr{V}_0$ is shown if the gradients are uniformly L^p-bounded. The latter, however, follows from (3.24), (3.25), (3.26) by chosing δ_k, $k = 1, 2, 3$, sufficiently small which verifies (3.23).

Let $\mathscr{C} \subset \mathscr{S}$ be any well-orderd chain. Then due to (3.23) it is norm-bounded in $\mathscr{V}_0$ which implies due to [9, Lemma 4.1.2] the existence of a nondecreasing sequence (u_n) of $\mathscr{C}$ converging to $u_s = \sup_{u \in \mathscr{C}} u$ as

$$u_n \rightharpoonup u_s \quad \text{(weakly) in } \mathscr{V}_0, \qquad \text{and} \qquad u_n \to u_s \quad \text{(strongly) in } L^p(\Omega), \tag{3.27}$$

Moreover, in view of the compactness of the trace operator we have

$$\gamma u_n \to \gamma u_s \qquad \text{(strongly) in } L^p(\Gamma). \tag{3.28}$$

To prove that the upper bound u_s of $\mathscr{C}$ belongs to $\mathscr{S}$, note that for all n the $u_n \in \mathscr{S}$ satisfy

$$\langle Au_n + Fu_n, \varphi \rangle + \int_\Gamma w_{1,n} \gamma\varphi \, d\Gamma = \langle h, \varphi \rangle, \qquad \varphi \in \mathscr{V}_0, \tag{3.29}$$

where $w_{1,n}(x) \in \partial j_1(\gamma u_n(x))$ a.e. on Γ. Due to the growth condition imposed on ∂j_k, $k = 1, 2$, and applying (3.23) the sequence $(w_{1,n})$ is bounded in $L^q(\Gamma)$ such that (by eventually passing to a subsequence again denoted by $(w_{1,n})$) it is weakly convergent in $L^q(\Gamma)$ to $w_{1,s}$ which yields

$$\int_\Gamma w_{1,n} \gamma\varphi \, d\Gamma \to \int_\Gamma w_{1,s} \, \gamma\varphi \, d\Gamma \qquad \text{as} \quad n \to \infty, \tag{3.30}$$

where $w_{1,s}(x) \in \partial j_1(\gamma u_s(x))$ for a.e. $x \in \Gamma$. Furthermore, with the special test function $\varphi = u_n - u_s$ from (3.29) we get

$$\langle Au_n + Fu_n, u_n - u_s \rangle = -\int_\Gamma w_{1,n} \, \gamma(u_n - u_s) \, d\Gamma + \langle h, u_n - u_s \rangle. \tag{3.31}$$

By means of the convergence properties (3.27), (3.28), and the boundedness of $(w_{1,n})$ the right-hand side of (3.31) tends to zero as $n \to \infty$ such that we have

$$\langle (A+F)\, u_n, u_n - u_s \rangle \to 0 \qquad \text{as} \quad n \to \infty,$$

which implies due to the pseudomonotonicity of the operator $A+F$ that the following convergence holds,

$$(A+F)\, u_n \rightharpoonup (A+F)\, u_s \qquad \text{in} \quad \mathscr{V}_0^*, \qquad \text{as} \quad n \to \infty, \tag{3.32}$$

cf., e.g., [12]. Using (3.30) and (3.32) we may pass to the limit as $n \to \infty$ in (3.29) which proves that $u_s \in \mathscr{S}$. Thus Zorn's lemma can be applied to the set $\mathscr{S}$, which ensures the existence of a *maximal element* u_{max} with respect to the underlying partial ordering.

(d) Existence of the greatest solution $u^* \in \mathscr{S}$. According to part (b) the solution set $\mathscr{S}$ is upward directed and by (c) there exists a maximal element in $\mathscr{S}$. This implies that the maximal element must be uniquely defined and must be the greatest one, i.e., $u_{max} = u^*$, which completes the proof of Lemma 3.1, since the proof for the existence of the least element u_* can be done by obvious dual reasoning. ∎

Remark 3.1. For (3.32) in part (c) of the proof of Lemma 3.1 we have used the following equivalent definition of pseudomonotonicity of an operator $\mathscr{A}: X \to X^*$ mapping a reflexive Banach space X into its dual X^*, cf., e.g., [12]: $\mathscr{A}$ is pseudomonotone, if for any sequence (u_n) in X with $u_n \rightharpoonup u$ and $\limsup \langle \mathscr{A}(u_n), u_n - u \rangle \leqslant 0$, it follows that $\mathscr{A}(u_n) \rightharpoonup \mathscr{A}(u)$ in X^* and $\langle \mathscr{A}(u_n), u_n \rangle \to \langle \mathscr{A}(u), u \rangle$.

4. PROOF OF THEOREM 2.1

Based on Lemma 3.1 we are going to show the existence of the greatest solution of the hemivariational inequality (1.6) within the order interval $[\underline{u}, \bar{u}]$ formed by the upper and lower solution.

Let $u_0 := \bar{u}$ and define $u_1 \in \mathscr{V}_0$ as the greatest solution within $[\underline{u}, u_0]$ of the variational inequality

$$\langle Au_1 + Fu_1, \varphi \rangle + \int_\Gamma w_{1,1} \gamma\varphi \, d\Gamma = \int_\Gamma \bar{H}_2(\gamma u_0)\, \gamma\varphi \, d\Gamma, \qquad \varphi \in \mathscr{V}_0, \tag{4.1}$$

where $w_{1,1}(x) \in \partial j_1(\gamma u_1(x))$. Since (4.1) corresponds with (3.1), the existence of such a greatest solution follows from Lemma 3.1. Let $u_2 \in \mathscr{V}_0$ be the greatest solution within $[\underline{u}, u_1]$ of the variational inequality

$$\langle Au_2 + Fu_2, \varphi \rangle + \int_\Gamma w_{1,2} \gamma\varphi \, d\Gamma = \int_\Gamma \bar{H}_2(\gamma u_1) \, \gamma\varphi \, d\Gamma, \qquad \varphi \in \mathscr{V}_0, \quad (4.2)$$

with $w_{1,2}(x) \in \partial j_1(\gamma u_2(x))$, which arises from (4.1) replacing u_0 of the right-hand side by u_1. Taking into account that $u_1 \in [\underline{u}, u_0]$ implies also $\gamma u_1 \in [\gamma\underline{u}, \gamma u_0]$, from (4.1) and the monotonicity of $\bar{H}_2$ it follows that u_1 is an upper solution for (4.2), and thus we have exactly the situation of Lemma 3.1 where u_1 takes over the role of $\bar{u}$. Hence by means of Lemma 3.1 the existence of a greatest solution $u_2 \in [\underline{u}, u_1]$ can be ensured. By induction we obtain the following well-defined iteration process: $u_0 := \bar{u}$ and $u_{n+1} \in \mathscr{V}_0$ is the greatest solution within $[\underline{u}, u_n]$ satisfying

$$\langle Au_{n+1} + Fu_{n+1}, \varphi \rangle + \int_\Gamma w_{1,n+1} \gamma\varphi \, d\Gamma = \int_\Gamma \bar{H}_2(\gamma u_n) \, \gamma\varphi \, d\Gamma, \qquad \varphi \in \mathscr{V}_0, \quad (4.3)$$

where $w_{1,n+1}(x) \in \partial j_1(\gamma u_{n+1}(x))$ a.e. in Γ.

The iteration (4.3) yields a nonincreasing sequence (u_n) that satisfies

$$\underline{u} \leqslant \cdots \leqslant u_{n+1} \leqslant u_n \leqslant \cdots \leqslant u_1 \leqslant u_0 = \bar{u}. \quad (4.4)$$

Similarly as in the proof of Lemma 3.1 one easily verifies that (u_n) is bounded in $\mathscr{V}_0$ which in view of (4.4) implies the following convergence properties:

(i) $u_n \rightharpoonup u^*$ (weakly) in $\mathscr{V}_0$,

(ii) $u_n \to u^*$ (strongly) in $L^p(\Omega)$,

(iii) $\gamma u_n \to \gamma u^*$ (strongly) in $L^p(\Gamma)$,

(iv) $w_{1,n} \rightharpoonup w_1^*$ (weakly) in $L^q(\Gamma)$ with $w_{1,n}(x) \in \partial j_1(\gamma u_n(x))$ a.e. in Γ.

Since the function $\bar{h}_2 \colon \mathbb{R} \to \mathbb{R}$ which generates the Nemytskij operator $\bar{H}_2$ is monotone nondecreasing and right-sided continuous we get by means of Lebesgue's dominated convergence theorem and taking the monotonicity of the sequence (u_n) into account

$$\int_\Gamma \bar{H}_2(\gamma u_n) \, \gamma\varphi \, d\Gamma \to \int_\Gamma \bar{H}_2(\gamma u^*) \, \gamma\varphi \, d\Gamma \qquad \text{as} \quad n \to \infty. \quad (4.5)$$

Further, from (4.3) and the convergence properties above we obtain

$$\langle Au_n + Fu_n, u_n - u^* \rangle = \int_\Gamma (-w_{1,n} + \bar{H}_2(\gamma u_{n-1}))(\gamma(u_n - u^*))\, d\Gamma \to 0,$$

as $n \to \infty$, which by means of the convergence properties (i)–(iv) and (4.5) along with the pseudomonotonicity of $A+F$ (see Remark 3.1) allows to pass to the limit $n \to \infty$ in (4.3). This shows that the limit $u^* \in \mathscr{V}_0$ satisfies

$$\langle Au^* + Fu^*, \varphi \rangle + \int_\Gamma w_1^*\, \gamma\varphi\, d\Gamma = \int_\Gamma \bar{H}_2(\gamma u^*)\, \gamma\varphi\, d\Gamma, \qquad \varphi \in \mathscr{V}_0, \tag{4.6}$$

where $w_1^*(x) \in \partial j_1(\gamma u^*(x))$ a.e. in Γ. Define $w_2^* := \bar{H}_2(\gamma u^*) \in L^q(\Gamma)$. Then $w_2^*(x) \in \partial j_2(\gamma u^*(x))$ for a.e. $x \in \Gamma$, which shows that the limit $u^* \in [\underline{u}, \bar{u}]$ is also a solution of the original hemivariational inequality (1.6) according to Definition 2.1.

We show that u^* is the greatest solution of (1.6) with respect to the interval $[\underline{u}, \bar{u}]$. First we are going to prove that the limit u^* of the iterates (u_n) is the greatest solution within the interval $[\underline{u}, \bar{u}]$ of the variational inequality

$$\langle Au + Fu, \varphi \rangle + \int_\Gamma w_1 \gamma\varphi\, d\Gamma = \int_\Gamma \bar{H}_2(\gamma u)\, \gamma\varphi\, d\Gamma, \qquad \varphi \in \mathscr{V}_0, \tag{4.7}$$

where $w_1(x) \in \partial j_1(\gamma u(x))$ a.e. in Γ. For this purpose let $\tilde{u} \in [\underline{u}, \bar{u}]$ be any solution of (4.7). Then $\tilde{u}$ is, in particular, a lower solution of (4.7) satisfying $\underline{u} \leqslant \tilde{u} \leqslant \bar{u}$. If we replace in the iteration scheme $\underline{u}$ by $\tilde{u}$ and start the iteration in the same way with $u_0 = \bar{u}$, then the iterate u_{n+1} defined as the greatest solution of (4.3) within the interval $[\tilde{u}, u_n]$ remains the same. Thus we get

$$\tilde{u} \leqslant \cdots \leqslant u_{n+1} \leqslant u_n \leqslant \cdots \leqslant u_0 = \bar{u}$$

which shows that $\tilde{u} \leqslant u^*$.

To prove that u^* is also greatest solution of the original problem (1.6) within the interval $[\underline{u}, \bar{u}]$ let $u \in [\underline{u}, \bar{u}]$ be any solution of (1.6) according to Definition 2.1. Then this solution must necessarily be a lower solution of (4.7) and by the same arguments as before we obtain $u \leqslant u_n$ and thus $u \leqslant u^*$. This completes the proof of Theorem 2.1, since the proof for the existence of a least solution in $[\underline{u}, \bar{u}]$ can be done similarly. ∎

Remark 4.1. The truncation technique used in the proof of Lemma 3.1 allows us to weaken the growth condition (H3) on f by the following one,

$$|f(x, s, \xi)| \leqslant k_0(x) + c_0\, |\xi|^{p-1},$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^N$ and for all $s \in [\underline{u}(x), \bar{u}(x)]$, where $k_0 \in L^q_+(\Omega)$ and $\bar{u}$, $\underline{u}$ are the given upper and lower solutions, respectively. In a similar way the growth condition (1.4) on $\partial_c j$ and ∂j_k can be relaxed by assuming only a $L^q(\Gamma)$-boundedness condition with respect to the order interval $[\gamma\underline{u}, \gamma\bar{u}]$.

Remark 4.2. From the proof of Theorem 2.1 it can be seen that the greatest solution u^* of the hemivariational inequality (1.6) is obtained as the greatest solution of the variational inequality (4.7) with respect to the order interval $[\underline{u}, \bar{u}]$. Similarly the least solution u_* of (1.6) within the order interval $[\underline{u}, \bar{u}]$ is obtained as the least solution within $[\underline{u}, \bar{u}]$ of the following variational inequality,

$$\langle Au + Fu, \varphi \rangle + \int_\Gamma w_1 \gamma\varphi \, d\Gamma = \int_\Gamma \underline{H}_2(\gamma u)\, \gamma\varphi \, d\Gamma, \qquad \varphi \in \mathscr{V}_0,$$

where $w_1(x) \in \partial j_1(\gamma u(x))$ a.e. in Γ.

5. APPLICATION: BOUNDARY SEMIPERMEABILITY PROBLEM

In various concrete applications it is not very hard to find upper and lower solutions by simple computations. Moreover, in some cases these upper and lower solutions can be shown to be global bounds for any solution of a particular problem, such that by our main result the existence of global extremal solutions can be ensured. To demonstrate the applicability of our results we consider a BVP from semipermeability theory.

Semipermeability problems were first considered in [7] for monotone semipermeability relations which lead to variational inequalities. They arise for instance in heat conduction, in electrostatics, and in flow problems through porous media. Later on also nonmonotone semipermeability relations have been taken into account which lead to hemivariational inequalities, since the potentials involved are nonconvex; cf., e.g., [11, 14, 16]. For the following model of a boundary semipermeability problem existence results have been obtained, e.g., [5, 11, 13]. Here we are going to prove the existence of *global extremal* solutions. We consider

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \setminus \Gamma, \qquad -\frac{\partial u}{\partial \nu} \in \partial_c j(u) \quad \text{on } \Gamma, \tag{5.1}$$

where Ω and Γ is as in the Introduction, $\partial/\partial v$ is the outer normal derivative, $f \in L^2(\Omega)$ is some given function, and $\partial_c j\colon \mathbb{R} \to 2^{\mathbb{R}} \setminus \varnothing$ is Clarke's generalized gradient of some locally Lipschitz function $j\colon \mathbb{R} \to \mathbb{R}$, whose graph is of the form

$$\partial_c j(s) = \begin{cases} 0 & \text{if} \quad s < s_0, \\ [-a, 0] & \text{if} \quad s = s_0, \\ \alpha(s - s_1) & \text{if} \quad s > s_0, \end{cases} \tag{5.2}$$

where $0 < s_0 < s_1$, $a > 0$, and $\alpha = a/(s_1 - s_0)$. Thus the generalized gradient given by (5.2) admits the representation

$$\partial_c j(s) = \partial j_1(s) - \partial j_2(s) \qquad \text{for all} \quad s \in \mathbb{R}, \tag{5.3}$$

where the subdifferentials $\partial j_k(s)$, $k = 1, 2$, are given by

$$\partial j_1(s) = \begin{cases} 0 & \text{if} \quad s \leqslant s_0, \\ \alpha(s - s_0) & \text{if} \quad s > s_0, \end{cases} \tag{5.4}$$

and

$$\partial j_2(s) = \begin{cases} 0 & \text{if} \quad s < s_0, \\ [0, a] & \text{if} \quad s = s_0, \\ a & \text{if} \quad s > s_0. \end{cases} \tag{5.5}$$

Let $\mathscr{V} = W^{1,2}(\Omega)$ and $\mathscr{V}_0 = \{v \in W^{1,2}(\Omega) \mid \gamma v = 0 \text{ on } \partial\Omega \setminus \Gamma\}$, then the mixed BVP (5.1) is equivalent to the following hemivariational inequality: Find $u \in \mathscr{V}_0$ such that

$$\langle Au, \varphi - u \rangle + \int_\Gamma j^o(\gamma u; \gamma\varphi - \gamma u)\, d\Gamma \geqslant \int_\Omega f(\varphi - u)\, dx, \qquad \text{for all} \quad \varphi \in \mathscr{V}_0, \tag{5.6}$$

where $A = -\Delta\colon \mathscr{V}_0 \to \mathscr{V}_0^*$ is linear and strongly monotone. We assume also that $\partial\Omega \setminus \Gamma$ has positive (surface) measure. The subdifferential $\partial j_1(s)$ is a singleton for each $s \in \mathbb{R}$, and both subdifferentials have at most a linear growth so that Hypotheses (H1) and (H2) are satisfied. Up to an additive constant the corresponding convex functions $j_k\colon \mathbb{R} \to \mathbb{R}$ whose subdifferentials are given by (5.4) and (5.5) can readily be found as

$$j_1(s) = 0 \qquad \text{for} \quad s \leqslant s_0 \qquad \text{and} \qquad j_1(s) = \frac{\alpha}{2}(s - s_0)^2 \qquad \text{for} \quad s > s_0,$$

$$j_2(s) = 0 \qquad \text{for} \quad s < s_0 \qquad \text{and} \qquad j_2(s) = a(s - s_0) \qquad \text{for} \quad s \geqslant s_0.$$

Obviously Hypotheses (A1)–(A3) and (H3) are trivially satisfied. Taking into account that ∂j_2 is uniformly bounded, i.e.,

$$\eta \in \partial j_2(s): 0 \leqslant \eta \leqslant a, \qquad \text{for all} \quad s \in \mathbb{R}$$

and $\partial j_1(s)$ is a singleton for each $s \in \mathbb{R}$ and thus j_1 is strictly differentiable, i.e., $\partial j_1(s) = \frac{d}{ds} j_1(s) = j_1'(s)$, an ordered pair of upper and lower solutions can be constructed as follows. Denote

$$\langle Au, \varphi \rangle = a(u, \varphi) := \int_\Omega \nabla u \nabla \varphi \, dx,$$

and consider the problems

$$\begin{aligned} u \in \mathscr{V}_0: \langle Au, \varphi \rangle + \int_\Gamma j_1'(\gamma u)\, \gamma\varphi \, d\Gamma \\ = a \int_\Gamma \gamma\varphi \, d\Gamma + \int_\Omega f\varphi \, dx, \qquad \text{for all} \quad \varphi \in \mathscr{V}_0, \end{aligned} \tag{5.7}$$

and

$$u \in \mathscr{V}_0: \langle Au, \varphi \rangle + \int_\Gamma j'_1(\gamma u)\, \gamma\varphi \, d\Gamma = \int_\Omega f\varphi \, dx, \qquad \text{for all} \quad \varphi \in \mathscr{V}_0. \tag{5.8}$$

Denoting the right-hand side of (5.7) and (5.8) by $\langle \bar{h}, \varphi \rangle$ and $\langle \underline{h}, \varphi \rangle$, respectively, then $\bar{h}, \underline{h} \in \mathscr{V}_0^*$. Note that if the measure of $\partial\Omega \setminus \Gamma$ is positive then an equivalent norm in $\mathscr{V}_0$ is given by

$$\|u\|_{\mathscr{V}_0}^2 = \int_\Omega |\nabla u|^2 \, dx. \tag{5.9}$$

Due to the monotonicity of j_1' and because of $j_1'(0) = 0$ the operator $L: \mathscr{V}_0 \to \mathscr{V}_0^*$ defined by the left-hand side of both (5.7) and (5.8) is strongly monotone and coercive. Obviously L is also continuous. Hence, by standard results from monotone operator theory each of the problems (5.7) and (5.8) has a unique solution. Denote by $\bar{u}$ and $\underline{u}$ the unique solution of (5.7) and (5.8), respectively. Then subtracting (5.7) from (5.8) we obtain for all nonnegative test function $\varphi \in \mathscr{V}_0 \cap L_+^2(\Omega)$ the inequality

$$\langle A\underline{u} - A\bar{u}, \varphi \rangle + \int_\Gamma (j_1'(\gamma\underline{u}) - j_1'(\gamma\bar{u}))\, \gamma\varphi \, d\Gamma \leqslant 0. \tag{5.10}$$

Using the special test function $\varphi=(\underline{u}-\bar{u})^+$ in (5.10) we get by means of the monotonicity of j'_1 the inequality

$$\int_\Omega |\nabla(\underline{u}-\bar{u})^+|^2\,dx \leqslant 0,$$

which yields $(\underline{u}-\bar{u})^+=0$ in Ω in view of (5.9), and thus

$$\underline{u}\leqslant \bar{u}. \tag{5.11}$$

Let $h_2(s\pm)$ be the one-sided limits introduced in Section 2 which generate $\partial j_2(s)$ by $\partial j_2(s)=[h_2(s-), h_2(s+)]$ and let $\bar{h}_2(s):=h_2(s+)$ and $\underline{h}_2(s):=h_2(s-)$ (see Section 2) as well as $\bar{H}_2$ and $\underline{H}_2$ their corresponding Nemytskij operators. Then by the uniform boundedness of ∂j_2 we get the inequality

$$0\leqslant \int_\Gamma \underline{H}_2(\gamma\underline{u})\,\gamma\varphi\,d\Gamma \leqslant \int_\Gamma \bar{H}_2(\gamma\bar{u})\,\gamma\varphi\,d\Gamma \leqslant a\int_\Gamma \gamma\varphi\,d\Gamma, \tag{5.12}$$

for all $\varphi\in\mathscr{V}_0\cap L^2_+(\Omega)$, which shows that $\bar{u}$ and $\underline{u}$ are upper and lower solution, respectively of the hemivariational inequality (5.6) according to Definition 2.2 and 2.3 with $\bar{w}_1=j'_1(\gamma\bar{u})$ and $\underline{w}_1=j'_1(\gamma\underline{u})$. By applying Theorem 2.1 there exist extremal solutions of the hemivariational inequality (5.6) within the interval $[\underline{u},\bar{u}]$. Next we are going to show that these extremal solutions are in fact global extremal ones. Let u be any solution of (5.6) wich means that $u\in\mathscr{V}_0$ and that there are $w_k\in L^2(\Gamma)$ satisfying $w_k(x)\in\partial j_k(\gamma u(x))$, $k=1, 2$, such that

$$\langle Au,\varphi\rangle+\int_\Gamma w_1\,\gamma\varphi\,d\Gamma=\int_\Gamma w_2\,\gamma\varphi\,d\Gamma+\int_\Omega f\varphi\,dx, \qquad \text{for all}\quad \varphi\in\mathscr{V}_0. \tag{5.13}$$

Since $\partial j_1=j'_1$, we have $w_1(x)=j'_1(\gamma u(x))$. We compare the upper solution $\bar{u}$ which is the unique solution of (5.7) with any solution of (5.6). For this purpose we subtract (5.7) from (5.13) and obtain due to the uniform boundedness of ∂j_2 the inequality

$$\langle Au-A\bar{u},\varphi\rangle+\int_\Gamma (j'_1(\gamma u)-j'_1(\gamma\bar{u}))\,\gamma\varphi\,d\Gamma\leqslant 0, \quad \text{for all}\ \ \varphi\in\mathscr{V}_0\cap L^2_+(\Omega).$$

Taking the special test function $\varphi=(u-\bar{u})^+$ in the last inequality we get in just the same way as above that $(u-\bar{u})^+=0$, and thus $u\leqslant\bar{u}$. The proof for $\underline{u}\leqslant u$, where $\underline{u}$ is the unique solution of (5.8) follows the same line. This completes the proof of the existence of global extremal solutions for (5.6).

Remark 5.1. As has been proved above the unique solution $\bar{u}$ of (5.7) and $\underline{u}$ of (5.8) are upper and lower solutions of the hemivariational inequality (5.6), respectively, satisfying $\underline{u} \leqslant \bar{u}$, and any solution of (5.6) including the extremal ones are contained in the interval $[\underline{u}, \bar{u}]$. Moreover, by comparison it can easily be seen that any upper solution $\bar{v}$ of (5.7) and any lower solution $\underline{v}$ of (5.8) are also upper and lower solutions of (5.6), respectively, satisfying

$$\underline{v} \leqslant \underline{u} \leqslant \bar{u} \leqslant \bar{v}.$$

In applications, however, it is a much easier task to get bounds for the solutions of (5.6) in the form of $\bar{v}$, $\underline{v}$ rather than to solve the variational inequalities (5.7) and (5.8) to get the sharper bounds $\bar{u}$ and $\underline{u}$, respectively. For example, let the function f of (5.1) be nonnegative. Then an upper solution $\bar{v}$ of (5.7) can immediately be given by $\bar{v}(x) = \bar{s} + w(x)$, where $\bar{s} > 0$ is any constant satisfying $j_1'(\bar{s}) \geqslant a$ and w is any upper solution of the problem

$$-\Delta u = f \qquad \text{in } \Omega, \qquad u = 0 \qquad \text{on } \partial\Omega \backslash \Gamma, \qquad \frac{\partial u}{\partial v} = 0 \qquad \text{on } \Gamma.$$

Obviously $\underline{v} \equiv 0$ is a lower solution of (5.8).

Remark 5.2. According to Remark 4.2 the greatest solution u^* of (5.6) is obtained as the greatest solution of the following problem: Find $u \in \mathscr{V}_0$ such that

$$\int_\Omega \nabla u \, \nabla \varphi \, dx + \int_\Gamma j_1'(\gamma u) \, \gamma\varphi \, d\Gamma = \int_\Gamma \bar{h}_2(\gamma u) \, \gamma\varphi \, d\Gamma + \int_\Omega f\varphi \, dx, \qquad \varphi \in \mathscr{V}_0, \tag{5.14}$$

where $\bar{h}_2 : \mathbb{R} \to \mathbb{R}$ is the right-side limit of the function h_2 generating the subdifferential ∂j_2 which is given by

$$\bar{h}_2(s) = 0 \qquad \text{if} \quad s < s_0, \qquad \bar{h}_2(s) = a \qquad \text{if} \quad s \geqslant s_0.$$

Moreover, u^* can be constructed by means of (5.14) using the iteration: $u_0 := \bar{v}$,

$$\int_\Omega \nabla u_{n+1} \, \nabla \varphi \, dx + \int_\Gamma j_1'(\gamma u_{n+1}) \, \gamma\varphi \, d\Gamma$$

$$= \int_\Gamma \bar{h}_2(\gamma u_n) \, \gamma\varphi \, d\Gamma + \int_\Omega f\varphi \, dx, \qquad \varphi \in \mathscr{V}_0, \tag{5.15}$$

where $\bar{v}$ may be any upper solution of (5.7). Similarly the least solution u_* of (5.6) is obtained as the least solution of (5.14) with $\bar{h}_2$ replaced by the left-side limit $\underline{h}_2$ which is given by

$$\underline{h}_2(s)=0 \qquad \text{if} \quad s \leqslant s_0, \qquad \underline{h}_2(s)=a \qquad \text{if} \quad s > s_0.$$

Existence results for the Neumann problem with discontinuous nonlinearities on the boundary $\partial\Omega$ having jumps upward but which need not necessarily one-sided continuous have been obtained in [3].

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